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Involution Schubert-Coxeter combinatorics

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Abstract. Suppose $K \subseteq \mathrm{GL}(n,\mathbb{C})$ is a closed subgroup which acts on the complete flag variety with finitely many orbits. When K is a Borel subgroup, these orbits are Schubert cells, whose study leads to Schubert polynomials and many connections to type A Coxeter combinatorics. When K is $\mathrm{O}(n,\mathbb{C})$ or $\mathrm{Sp}(n,\mathbb{C})$, the orbits are indexed by some involutions in the symmetric group. Wyser and Yong described polynomials representing the cohomology classes of the orbit closures, and we investigate parallels for these "involution Schubert polynomials" of classical combinatorics surrounding type A Schubert polynomials. We show that their stable versions are Schur-P-positive, and obtain as a byproduct a new Littlewood-Richardson rule for Schur P-functions.

A key tool is an analogue of weak Bruhat order on involutions introduced by Richardson and Springer. This order can be defined for any Coxeter group W, and its labelled maximal chains correspond to reduced words for distinguished elements of W which we call *atoms*. In type A we classify all atoms, generalizing work of Can, Joyce, and Wyser, and give a connection to the *Chinese monoid* of Cassaigne et al. We give a different description of some atoms in general finite W in terms of strong Bruhat order.

Résumé. Soit $K \subseteq \mathrm{GL}(n,\mathbb{C})$ un sous-groupe fermé qui agit sur la variété de drapeaux complets avec un nombre fini d'orbites. Quand K est un sous-groupe de Borel, ces orbites sont des cellules de Schubert, dont l'étude mène aux polynômes de Schubert et de nombreuses connexions aux combinatoires de Coxeter de type A. Quand K est $\mathrm{O}(n,\mathbb{C})$ ou $\mathrm{Sp}(n,\mathbb{C})$, les orbites sont indexées par des involutions du groupe symétrique. Wyser et Yong ont décrit des polynômes représentant les classes de cohomologie des clôtures de ces orbites, et nous étudions des parallèles pour ces polynômes de la combinatoire classique des polynômes de Schubert de type A. Nous montrons que les versions stables de ces polynômes sont Schur-P-positives, et nous obtenons par ce biais une nouvelle règle de Littlewood-Richardson pour les P-fonctions de Schur.

Un outil clé est l'analogue de l'ordre faible de Bruhat sur les involutions introduit par Richardson et Springer. Cet ordre peut être défini pour tout groupe de Coxeter W, et ses chaînes maximalles étiquetées correspondent aux mots réduits pour des éléments spéciaux de W que nous appelons *atomes*. Pour le type A, nous classifions tous les atomes, généralisant les travaux de Can, Joyce et Wyser, et donnant une connexion au *monoïde chinois* de Cassaigne et al. Nous donnons une des description différente de certains atomes pour un W fini général en termes d'un ordre fort de Bruhat.

Keywords: Schubert polynomials, Coxeter combinatorics, spherical orbits

1 Introduction

Let (W, S) be a Coxeter system, and $*$ an involutive graph automorphism of the associated Coxeter diagram. We will call $(W, S, *)$ a *twisted Coxeter system*. The automorphism $*$ induces an involution $* : W \to W$, whose action we denote $x \mapsto x^*$. The set of *twisted involutions* of $(W, S, *)$ is $\mathcal{I}_*(W) := \{x \in W : x^{-1} = x^*\}$. When $*$ is the identity we will simply write $\mathcal{I}(W)$.

When W is the Weyl group of a connected reductive linear algebraic group G with Borel subgroup B and maximal torus T, twisted involutions arise in the following geometric setting studied by Richardson and Springer [20]. Suppose $\theta : G \to G$ is an involutive automorphism stabilizing T and B (hence acting on W), and let $K \subseteq G$ be the *symmetric subgroup* of θ-fixed points. Then K acts on the flag variety G/B with finitely many orbits, and Richardson and Springer defined a map from the set of K-orbits on G/B into the set of twisted involutions $\{x \in W : x^{-1} = \theta(x)\}$.

The study of the Schubert varieties in G/B—that is, the B-orbit closures, indexed by W—has been a rich source of problems in algebraic combinatorics. For instance, Bruhat order, symmetric functions, Schubert polynomials, and Kazhdan-Lusztig polynomials are all intimately related to geometric questions about Schubert varieties. The machinery of the previous paragraph then suggests a program of investigating analogues of classical Schubert combinatorics for twisted Coxeter systems of Weyl groups in connection with the relevant geometry of K-orbits on G/B.

Alternatively, one can simply approach the study of twisted involutions as a problem in Coxeter theory, as in [16]. Richardson and Springer introduced the *(twisted) involution weak order* on $\mathcal{I}_*(W)$, which resembles weak order on W. The labelled maximal chains in an interval $[x, y]$ of this poset, which we call *involution words*, behave in many ways like reduced words for a Coxeter group element. In fact, these involution words are in bijection with reduced words for a certain subset of W associated to $[x, y]$, the set of *atoms* of $[x, y]$.

We will describe results coming from both directions. In Section 2, we give some background on twisted involutions in a general Coxeter group, and outline our results on atoms. More specifically, we generalize work in [6] to classify all atoms in type A, and note an unexpected connection to the *Chinese monoid* of [7]. We also give a new and different characterization of atoms for $[1, x]$ in any finite W in terms of strong Bruhat order on W, and conjecture its correctness for all intervals in all W.

Section 3 focuses on questions arising from the geometry of K-orbits on the type A flag variety where K is an orthogonal or symplectic group. Analogues of Schubert polynomials, which represent the cohomology classes of these K-orbit closures, were found in [3] and [22]. We give some new identities for these *involution Schubert polynomials*. In particular, we find an analogue of Lascoux and Schützenberger's transition formula for ordinary Schubert polynomials [18].

In Section 4 we consider enumerative problems for involution words. We prove using the aforementioned transition formula that (in the case $K = \mathrm{O}(n)$) stable involution Schubert polynomials are not only Schur-positive, but Schur-P-positive. Their Schur-P expansions yield enumerations of involution words in terms of shifted standard Young tableaux, which can be proven bijectively using an insertion algorithm introduced in [14]. These connections lead to a new proof of a conjecture of Stanley on Schur-P-positivity of skew Schur functions (proven by other means in [1] and [8]), and a new Littlewood-Richardson rule for Schur-P-functions.

2 Involution words and atoms

Let $(W, S, *)$ be a twisted Coxeter system, with length function ℓ. Given $s \in S$ and $w \in W$, define

$$w \circ s = \begin{cases} ws & \text{if } \ell(w) < \ell(ws) \\ w & \text{if } \ell(w) > \ell(ws) \end{cases}$$

The operation $\circ$ extends to an associative binary operation on W; up to a sign convention, this is multiplication in the 0-Hecke algebra of W.

Definition 1. Let $\hat{\circ}_*$ be the binary operation on W defined by $x \mathbin{\hat{\circ}_*} w = (w^*)^{-1} \circ x \circ w$. The *(twisted involution) weak order* on $\mathcal{I}_*(W)$ is the order $<_{\mathcal{I}}^*$ defined by the cover relations $x \lessdot_{\mathcal{I}}^* x \mathbin{\hat{\circ}_*} s$ if $x < x \mathbin{\hat{\circ}_*} s$.

Note that $\hat{\circ}_*$ is not associative. The following simple formula is useful.

Proposition 2. *For $x \in \mathcal{I}_*(W)$ and $s \in S$,*

$$x \mathbin{\hat{\circ}_*} s = \begin{cases} x & \text{if } xs < x \\ xs & \text{if } x < xs = s^*x \\ s^*xs & \text{if } x < xs \neq s^*x \end{cases}.$$

Definition 3. Take $x, y \in \mathcal{I}_*(W)$. An *involution word* of y with respect to x is a minimal-length sequence $(s_1, \ldots, s_p)$ of simple generators such that

$$y = (\cdots((x \mathbin{\hat{\circ}_*} s_1) \mathbin{\hat{\circ}_*} s_2) \mathbin{\hat{\circ}_*} \cdots) \mathbin{\hat{\circ}_*} s_p.$$

Let $\hat{\mathcal{R}}_*(x, y)$ denote the set of involution words of y with respect to x.

An involution word is thus a labelling of a maximal chain in the involution weak order on $[x, y]$ (but note that distinct involution words can correspond to the same maximal chain, as in Example 6 below). Involution weak order and involution words were introduced in [20], where the latter are called "admissible sequences". In the usual weak order on W, the interval $[v, w]$ is isomorphic to $[1, v^{-1}w]$, but no such statement holds in the involution setting, so the study of arbitrary intervals becomes interesting.

Definition 4. The sets of *Hecke atoms* and *atoms*, respectively, of y relative to x are

$$\mathcal{B}_*(x,y) = \{w \in W : x \mathbin{\hat{\circ}}_* w = y\} \quad \text{and} \quad \mathcal{A}_*(x,y) = \{w \in \mathcal{B}_*(x,y) : \ell(w) \text{ is minimal}\}.$$

Write $\mathcal{R}(w)$ for the set of reduced words of $w \in W$.

Proposition 5. *$\hat{\mathcal{R}}(x,y) = \bigcup_{w\in\mathcal{A}(x,y)} \mathcal{R}(w)$ for any $x, y \in \mathcal{I}_*(W)$.*

Example 6. With $W = S_4$ and $* = \mathrm{id}$,

$$\begin{aligned} &\mathcal{R}_*(1,(1\,3)) = \{(s_1,s_2),(s_2,s_1)\} && \text{so} \quad \mathcal{A}_*(1,(1\,3)) = \{2314, 3124\}, \\ &\mathcal{R}_*((1\,2)(3\,4),(1\,4)(2\,3)) = \{(s_2,s_1),(s_2,s_3)\} && \text{so} \quad \mathcal{A}_*((1\,2)(3\,4),(1\,4)(2\,3)) = \{3124, 1342\}. \end{aligned}$$

The second example shows that distinct involution words can correspond to the same saturated chain in involution weak order.

2.1 Bruhat characterization of atoms

Write $\hat{\ell}_*(y)$ for the common length of the elements of $\hat{\mathcal{R}}_*(1,y)$, which is non-empty by Proposition 5. More generally, elements of $\hat{\mathcal{R}}_*(x,y)$ have length $\hat{\ell}_*(y) - \hat{\ell}_*(x)$.

For $w \in \mathcal{A}_*(x,y)$, Proposition 2 and the subword criterion for Bruhat order imply that $y = (w^*)^{-1}xw'$ where $w' \leq w$ and the product is length-additive, so that $w^*y \leq xw$. We conjecture that this condition in fact characterizes atoms among $w \in W$ with the correct length.

Conjecture 7. *For any $x, y \in \mathcal{I}_*(W)$,*

$$\mathcal{A}_*(x,y) = \{w \in W : w^*y \leq xw \text{ and } \ell(w) = \hat{\ell}_*(y) - \hat{\ell}_*(x)\}.$$

When W is finite and either x or y has a special form, we can prove Conjecture 7 by downwards induction on length. In particular, the conjecture holds when $x = 1$, as well as in the important special case where $W = S_{2n}$ and $x = (1\,2)(3\,4)\cdots(2n{-}1\ 2n)$.

Theorem 8 ([12])**.** *Suppose W is finite with longest element w_0, and $x, y \in \mathcal{I}_*(W)$ are such that $\ell(x) = \hat{\ell}_*(x)$ or $\ell(w_0) - \ell(y) = \hat{\ell}_*(w_0) - \hat{\ell}_*(y)$. Then Conjecture 7 holds.*

2.2 Atoms in type A

We now restrict to the case $W = S_n$ and $* = \mathrm{id}$, where we can give a complete characterization of atoms in a rather different way, extending the approach initiated in [4] and [6]. Define the *cycle set* and *extended cycle set* of $y \in \mathcal{I}(S_n)$ by

$$\begin{aligned} \mathrm{Cyc}(y) &= \{(a,b) : 1 \leq a \leq b \leq n \text{ and } y(a) = b\} \\ \mathrm{exCyc}(y) &= \mathrm{Cyc}(y) \cup \{(a,b) : 1 \leq b < a \leq n \text{ and } y(a) = a,\ y(b) = b\}. \end{aligned}$$

First we note a characterization of the sets $\mathcal{A}(1,y) = \mathcal{A}_{\mathrm{id}}(1,y)$ due to Can, Joyce, and Wyser.

Theorem 9. *[6], [12, Corollary 5.13] For $y \in \mathcal{I}(S_n)$, we have $w \in \mathcal{A}(1, y)$ if and only if:*

(a) Whenever $(a, b) \in \mathrm{Cyc}(y)$ it holds that $w(b) \leq w(a)$, and there is no $a < t < b$ with $w(b) < w(t) < w(a)$.

(b) Whenever $(a, b), (a', b') \in \mathrm{Cyc}(y)$ satisfy $a < a'$ and $b < b'$, it holds that $w(b) \leq w(a) < w(b') \leq w(a')$.

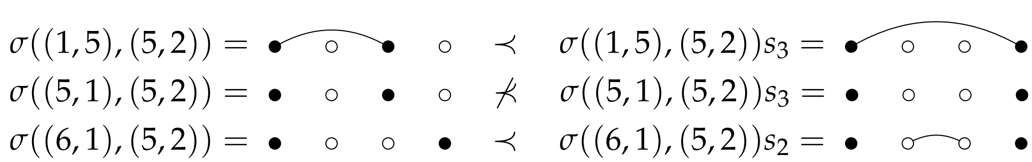
It is possible to give a similarly explicit description of $\mathcal{A}(x, y)$ for any $x, y \in \mathcal{I}(S_n)$ (see [12, Theorem 5.11]), but the inequalities and possible interactions between pairs of cycles become more complicated and less memorable. To capture these interactions in a more compact way we introduce the following bookkeeping device.

Definition 10. A *colored involution* on $[2n] = \{1, 2, \ldots, 2n\}$ is a partial matching of $[2n]$ whose vertices are colored by n colors, such that (a) each color appears exactly twice, and (b) if i and j are connected, they have the same color. Let $\mathcal{CI}_{2n}$ denote the set of colored involutions on $[2n]$.

To an ordered pair $((a_1, a_2), (a_3, a_4))$ of pairs of integers, we associate a colored involution $\sigma((a_1, a_2), (a_3, a_4)) \in \mathcal{CI}_4$ as follows. List the integers a_1, a_2, a_3, a_4 in weakly increasing order, identifying them with $1, 2, 3, 4$; if $a_i = a_j$ with $i < j$, view a_i as coming before a_j in the list. Color a_1, a_2 black, and a_3, a_4 white. Connect the pair a_1, a_2 if $a_1 < a_2$, and likewise for a_3, a_4.

A simple transposition $(i\ i{+}1)$ acts on $\alpha \in \mathcal{CI}_{2n}$ by (a) swapping vertices i and $i + 1$ if they have different colors, or (b) changing whether or not i and $i + 1$ are connected if they have the same color. Let $\pi : \mathcal{CI}_{2n} \to \mathcal{I}(S_{2n})$ be the map which forgets colors, and let $\prec$ be the weakest partial order on $\mathcal{CI}_{2n}$ such that $\alpha \prec \alpha s$ whenever $\pi(\alpha) <_{\mathcal{I}} \pi(\alpha s)$.

Example 11.



Given $(a, b), (c, d) \in [n] \times [n]$ and $w \in S_n$, write $(a, b) \cap (c, d)$ for $\{a, b\} \cap \{c, d\}$, and $w(a, b)$ for $(w(a), w(b))$. We can now state our characterization of atoms in type A.

Theorem 12 ([12], Theorem 5.10)**.** *For $x, y \in \mathcal{I}(S_n)$, we have $w \in \mathcal{A}(x, y)$ if and only if:*

(a) $w\gamma \in \mathrm{exCyc}(x)$ for all $\gamma \in \mathrm{Cyc}(y)$;

(b) $\sigma(w\gamma, w\gamma') \preceq \sigma(\gamma, \gamma')$ for all $\gamma, \gamma' \in \mathrm{Cyc}(y)$ with $\gamma \cap \gamma' \neq 0$.

The fact that this theorem requires only "local" checks—that is, conditions on pairs of cycles γ, γ' which depend only on the relative orders of the integers in γ, γ' and $w\gamma, w\gamma'$—opens the door to its use in computer proofs, as for Theorem 24.

2.3 Type A atoms and the Chinese monoid

Having fixed n, define $w_{\mathrm{FPF}} = (1\,2)(3\,4)\cdots(2n{-}1\;2n)$ in $\mathcal{I}(S_{2n})$, and

$$\mathcal{I}_{\mathrm{FPF}}(S_{2n}) = \{z \in \mathcal{I}(S_{2n}) : z \text{ has no fixed points}\} = \{z \in \mathcal{I}(S_{2n}) : z \geq_{\mathcal{I}} w_{\mathrm{FPF}}\}.$$

In this section we adopt some simpler notation, writing $\mathcal{A}(z)$ for $\mathcal{A}(1,z)$ and $\mathcal{R}(z)$ for $\mathcal{R}(1,z)$, and likewise $\mathcal{A}_{\mathrm{FPF}}(z)$ for $\mathcal{A}(w_{\mathrm{FPF}},z)$ and $\mathcal{R}_{\mathrm{FPF}}(z)$ for $\mathcal{R}(w_{\mathrm{FPF}},z)$. Recall also the notion of Hecke atoms $\mathcal{B}(z)$ from Definition 4.

Definition 13 ([7])**.** The *Chinese monoid* is the free monoid on $\mathbb{N}$ modulo the relations $bca \equiv cab \equiv cba$ where $a < b < c$, applied to any three consecutive letters in a word.

Theorem 14 ([12], Theorem 6.4)**.** *The Chinese monoid equivalence classes in S_n are exactly the sets $\mathcal{B}(z)^{-1} = \{w^{-1} : w \in \mathcal{B}(z)\}$ for $z \in \mathcal{I}(S_n)$.*

By focusing on the length-preserving relation $bca \equiv cab$ one obtains an alternate description of (inverse) atoms in type A. Equip $\mathcal{A}(z)^{-1}$ with a poset structure having a cover relation $w \lessdot w'$ whenever w' is obtained from w by replacing a consecutive subsequence cab by bca when $a < b < c$.

Theorem 15. *The poset $\mathcal{A}(z)^{-1}$ is graded and has unique minimal and maximal elements. Moreover, if $\mathrm{Cyc}(w) = \{(a_1, b_1), \ldots, (a_k, b_k)\}$ where $a_1 < \cdots < a_k$, then the minimal element of $\mathcal{A}(z)^{-1}$ has one-line notation $b_1 a_1 \cdots b_k a_k$ with a_i omitted when $a_i = b_i$.*

Recall that for a fixed $w \in S_n$, the reduced words $\mathcal{R}(w)$ form a single equivalence class under the Coxeter relations $s_i s_j = s_j s_i$ if $|i - j| > 1$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. Theorem 15 implies an analogous result for involution words, proven independently by Hu and Zhang [15]; we also obtain such a result for fixed-point-free involution words via a fixed-point-free version of Theorem 15.

Theorem 16 ([12], Theorem 7.1; [15])**.** *For a fixed $z \in \mathcal{I}(S_n)$, the involution words $\hat{\mathcal{R}}(z)$ form a single equivalence class under the Coxeter relations together with the relation $(s_i, s_j, \ldots) \sim (s_j, s_i, \ldots)$ for any $i, j \in [n-1]$.*

Theorem 17 ([12], Theorem 7.2)**.** *For a fixed $z \in \mathcal{I}_{\mathrm{FPF}}(S_{2n})$, the involution words $\hat{\mathcal{R}}_{\mathrm{FPF}}(z)$ form a single equivalence class under the Coxeter relations together with the relation $(s_{2i}, s_{2i-1}, \ldots) \sim (s_{2i}, s_{2i+1}, \ldots)$ for any $i \in [n-1]$.*

3 Involution Schubert calculus

In this section it will be helpful to write $\mathcal{I}_K$ for $\mathcal{I}(S_n)$ or $\mathcal{I}_{\mathrm{FPF}}(S_n)$ depending on whether $K = \mathrm{O}(n, \mathbb{C})$ or $K = \mathrm{Sp}(n, \mathbb{C})$, and likewise $\mathcal{A}_K(z)$, $\hat{\mathcal{R}}_K(z)$, etc. In any statement about $\mathrm{Sp}(n)$ it should be assumed that n is even. Let $\mathrm{Fl}(n)$ be the variety of complete flags in $\mathbb{C}^n$, with its right action of $\mathrm{GL}(n, \mathbb{C})$. The first of the following claims is well-known, while the second is proven in [20]:

- The orbits of the Borel group of upper triangular matrices on $\mathrm{Fl}(n)$ are in bijection with S_n.
- For $K = \mathrm{O}(n)$ or $\mathrm{Sp}(n)$, the K-orbits on $\mathrm{Fl}(n)$ are in bijection with $\mathcal{I}_K$.

These orbits can be described explicitly as follows. For $w \in S_n$, let $\mathrm{rk}_w(i,j) = \#\{p \in [i] : w(p) \in [j]\}$. The *rank* of a bilinear form $\alpha : V \times W \to \mathbb{C}$ is the rank of the matrix $[\alpha(v,w)]$ where v and w run over bases for V and W. Let $\mathrm{prj}_j : \mathbb{C}^n \to \mathbb{C}^j$ be projection onto the first j coordinates, and fix a non-degenerate bilinear form α^K on $\mathbb{C}^n$ which is symmetric if $K = \mathrm{O}(n)$ and skew-symmetric if $K = \mathrm{Sp}(n)$. Now for $w \in S_n$ and $z \in \mathcal{I}_K$, define

$$\mathring{X}_w = \{F_\bullet \in \mathrm{Fl}(n) : \mathrm{rank}\, \mathrm{prj}_j|_{F_i} = \mathrm{rk}_w(i,j) \text{ for all } i,j \in [n]\}$$
$$\mathring{Y}^K_z = \{F_\bullet \in \mathrm{Fl}(n) : \mathrm{rank}\, \alpha^K|_{F_i \times F_j} = \mathrm{rk}_z(i,j) \text{ for all } i,j \in [n]\}.$$

The sets $\mathring{X}_w$ are the classical type A *Schubert cells*, their closures X_w being *Schubert varieties*, and it is a basic fact that they are the B-orbits on $\mathrm{Fl}(n)$. It is shown in [21] that the sets $\mathring{Y}^K_z$ are indeed the orbits of $K = \mathrm{O}(n)$ or $\mathrm{Sp}(n)$ as z ranges over $\mathcal{I}_K$. Let Y^K_z be the Zariski closure of $\mathring{Y}^K_z$.

For a subvariety $X \subseteq \mathrm{Fl}(n)$, let $[X] \in H^*(\mathrm{Fl}(n), \mathbb{Z})$ denote the cohomology class Poincaré dual to X. Let $\kappa(z)$ denote the number of 2-cycles of an involution z.

Theorem 18 ([3]). *For $z \in \mathcal{I}_K$,*

$$[Y^K_z] = \begin{cases} 2^{\kappa(z)} \sum_{w \in \mathcal{A}_K(z)} [X_w] & \text{if } K = \mathrm{O}(n) \\ \sum_{w \in \mathcal{A}_K(z)} [X_w] & \text{if } K = \mathrm{Sp}(n) \end{cases}$$

The cohomology ring $H^*(\mathrm{Fl}(n), \mathbb{Z})$ is isomorphic to a quotient of $\mathbb{Z}[x_1, \ldots, x_n]$, and under this isomorphism the *Schubert polynomial* $\mathfrak{S}_w$ is a representative for $[X_w]$. See [19, Ch. 3] for an introduction to Schubert varieties and polynomials. In light of Theorem 18, we make the following definition.

Definition 19. The *involution Schubert polynomial* (for the group K) associated to $z \in \mathcal{I}_K$ is $\hat{\mathfrak{S}}^K_z = \sum_{w \in \mathcal{A}_K(z)} \mathfrak{S}_w$.

Thus $2^{\kappa(z)} \hat{\mathfrak{S}}^{\mathrm{O}(n)}_z$ and $\hat{\mathfrak{S}}^{\mathrm{Sp}(n)}_z$ are polynomial representatives for $[Y^{\mathrm{O}(n)}_z]$ and $[Y^{\mathrm{Sp}(n)}_z]$ respectively, by Theorem 18. Wyser and Yong [22] construct polynomial representatives for the classes $[Y^K_z]$ in a different way, using divided difference operators; the arguments in [22, §3.1] imply that their polynomials agree with ours, or see [10, §3.4].

3.1 Identities for involution Schubert polynomials

For various special classes of permutations $w \in S_n$, the classical Schubert polynomials have particularly nice forms. For instance, if w is dominant (avoids 132), then $\mathfrak{S}_w$ is a monomial; if w is Grassmannian (has at most one descent), then $\mathfrak{S}_w$ is a Schur polynomial. In this section we give involution analogues of these two facts.

Definition 20. An involution is *weakly dominant* if its disjoint cycle decomposition has the form $(1\,a_1)(2\,a_2)\cdots(k\,a_k)$, where $a_1, \ldots, a_k > k$.

If $z \in \mathcal{I}(S_n)$ is weakly dominant, let $r(z) \in S_n$ be the permutation with one-line notation $(a_1{-}k)\cdots(a_k{-}k)b_1\cdots b_{n-k}$, where $\{b_1 < \cdots < b_{n-k}\} = [n] \setminus \{a_1 - k, \ldots, a_k - k\}$. Let $g_k = (1\ k)(2\ k{+}1)\cdots(k\ 2k) \in S_n$.

Theorem 21 ([10], Theorem 3.27)**.** *If $z \in \mathcal{I}_K$ is weakly dominant with k 2-cycles, then*

$$\hat{\mathfrak{S}}^K_z = \hat{\mathfrak{S}}^K_{g_k}\mathfrak{S}_{r(z)}(x_1, \ldots, x_k, 0, 0, \ldots; -x_{k+1}, \ldots, -x_n, 0, 0, \ldots), \tag{3.1}$$

where $\mathfrak{S}_w(x_1, x_2, \ldots; y_1, y_2, \ldots)$ is the double Schubert polynomial of w.

When $z = w_n$ is the longest element of S_n, Theorem 21 recovers the following product formulas of Wyser and Yong [22] (see also the more general product formulas in [5]):

$$\hat{\mathfrak{S}}^K_{w_n} = \begin{cases} x_1\cdots x_n \prod_{1\le i<j\le n}(x_i + x_j) & \text{if } K = \mathrm{O}(n) \\ \prod_{1\le i<j\le n}(x_i + x_j) & \text{if } K = \mathrm{Sp}(n) \end{cases}$$

Definition 22. An *I-Grassmannian* involution is one with disjoint cycle decomposition $z = (\phi_1\ m{+}1)(\phi_2\ m{+}2)\cdots(\phi_k\ m{+}k)$ where $\phi_1 < \cdots < \phi_k < m$. We write $\hat{\mathfrak{S}}[\phi_1, \ldots, \phi_k; m]$ for $\hat{\mathfrak{S}}^{\mathrm{O}(n)}_z$.

Theorem 23. *Let $A[\phi_1, \ldots, \phi_k; m]$ be the $2\lceil \frac{k}{2} \rceil \times 2\lceil \frac{k}{2} \rceil$ skew-symmetric matrix with (i, j) entry $\hat{\mathfrak{S}}[\phi_i, \phi_j; m]$ for $i < j$, or $\hat{\mathfrak{S}}[\phi_i; m]$ if $j = k + 1$. Then $\hat{\mathfrak{S}}[\phi_1, \ldots, \phi_k; m]$ is the Pfaffian of $A[\phi_1, \ldots, \phi_k; m]$.*

One can view Theorem 23 as an analogue of the Jacobi-Trudi formula, which expresses the Schubert polynomial of a Grassmannian permutation as the determinant of a matrix filled with Schubert polynomials for single-row Grassmannians. There is a similar but slightly more complicated notion of "I-Grassmannian" for fixed-point-free involutions, and a corresponding Pfaffian formula.

3.2 Transition formulas and Bruhat order on involutions

Recall that $v \le w$ in strong Bruhat order if and only if some (every) reduced word for w has a subword in $\mathcal{R}(v)$. An involution version of this statement holds; namely, involutions y and z satisfy $y \leq z$ if and only if some (every) involution word for z has a subword in $\hat{\mathcal{R}}(y)$. Equivalently, $y \leq z$ if and only if for some (every) $w \in \mathcal{A}(z)$, there exists $v \in \mathcal{A}(y)$ such that $v \leq w$.

Theorem 24 ([11], Theorem 3.20)**.** *Fix $y \in \mathcal{I}(S_n)$ and $i < j$. Then there is at most one $z \in \mathcal{I}(S_n)$ such that for some $w \in \mathcal{A}(y)$, we have $w \lessdot w(i\,j)$ and $w(i\,j) \in \mathcal{A}(z)$.*

This gives a natural way of labelling the cover relations in $\mathcal{I}(S_n)$ equipped with Bruhat order: if there exists $w \in \mathcal{A}(y)$ such that $w(i\,j) \in \mathcal{A}(z)$ with $i < j$, define $\tau_{ij}(y) = z$, and otherwise set $\tau_{ij}(y) = y$. Note that it can happen that $\tau_{ij}(z) = \tau_{i'j'}(z)$ but $(i, j) \neq (i', j')$. Incitti [17] proved that the restriction of Bruhat order to $\mathcal{I}(S_n)$ is still graded and also labelled the cover relations by pairs of integers, albeit in a slightly different way.

Example 25. Let $y = (1\,2)(3\,4)$ and $y' = (1\,3)$. One checks that $w = 2143 \in \mathcal{A}(y)$ and $w' = 3124 \in \mathcal{A}(y')$, and that $\mathcal{A}((1\,4))$ contains $w(1\,4) = 3142$, $w'(1\,4) = 4123$, and $w(1\,3) = 4123$. Thus, $\tau_{14}(y) = \tau_{14}(y') = \tau_{13}(y) = (1\,4)$.

On the other hand, $w' \lessdot w'(2,3) = 3214$, and the latter is not an atom of any involution, so we set $\tau_{23}(y') = y'$. The first equality above presents an obstacle to defining a reasonable inverse of τ_{ij}.

Proof sketch of Theorem 24. Suppose that there is a counterexample to the theorem: $w, w' \in \mathcal{A}(y)$ such that $w \lessdot w(i\,j)$, $w' \lessdot w'(i\,j)$, and $w(i\,j) \in \mathcal{A}(z)$, $w'(i\,j) \in \mathcal{A}(z')$ with $z \neq z'$. Incitti's classification of involution Bruhat covers implies that that there is a set $E \subseteq [n]$ with $|E| \leq 8$ such that y, z, and z' stabilize E and agree on $[n] \setminus E$. The key point is now that the "local" nature of Theorem 12 classifying atoms immediately implies that the atom relationships in the counterexample still hold upon restricting w, w', y, z, z' to E and standardizing. One thereby deduces that any counterexample would lead to a counterexample in S_8, and a computer check rules out the latter. □

The restriction of Bruhat order to $\mathcal{I}_{\mathrm{FPF}}(S_n)$ is simpler to understand: if z covers y, then $z = (i\,j)y(i\,j)$ for some $i < j$ (see [11, §4.1]). We therefore define $\tau^K_{ij}(y)$ to be what we have called $\tau_{ij}(y)$ if $K = \mathrm{O}(n)$, and $(i\,j)y(i\,j)$ if $K = \mathrm{Sp}(n)$.

Given $y \in \mathcal{I}_K$ and $r \in \mathbb{N}$, define sets

$$\hat{\Phi}^+_K(y, r) = \{\tau^K_{rj}(y) : r < j \text{ and } \tau^K_{rj}(y) > y\}$$
$$\hat{\Phi}^-_K(y, r) = \{\tau^K_{ir}(y) : 1 \leq i < r \text{ and } \tau^K_{ir}(y) > y\}.$$

In this definition we identify a permutation $y \in S_n$ with the permutation of $\mathbb{N}$ agreeing with y on $[n]$ and fixing all $p > n$, so j need not be in $[n]$. The next theorem is an analogue of Lascoux and Schützenberger's transition formula for Schubert polynomials [18]:

Theorem 26 ([11], Theorems 3.28 and 4.17)**.** *For any $y \in \mathcal{I}_K$ and $q = y(p)$,*

$$2^{-\delta_{pq}}(x_p + x_q)\hat{\mathfrak{S}}^K_y = \sum_{z \in \hat{\Phi}^-_K(y,p)} \hat{\mathfrak{S}}^K_z - \sum_{z \in \hat{\Phi}^+_K(y,q)} \hat{\mathfrak{S}}^K_z.$$

4 Involution Stanley symmetric functions and enumerations of involution words

The *Stanley symmetric function* or *stable Schubert polynomial* of $w \in S_n$ is $F_w := \lim_{m\to\infty} \mathfrak{S}_{1^m \times w}$, where $1^m \times w = 12\cdots m(w(1)+m)\cdots(w(n)+m)$. It follows from [2] that $|\mathcal{R}(w)|$ is the coefficient of $x_1 x_2 \cdots x_{\ell(w)}$ in F_w, so from the Schur expansion $F_w = \sum_\lambda c_{\lambda,w} s_\lambda$, one deduces that $|\mathcal{R}(w)| = \sum_\lambda c_{\lambda,w} f^\lambda$, where f^λ is the number of standard tableaux of shape λ. The Edelman-Greene insertion algorithm [9] interprets the coefficients $c_{\lambda,w}$ bijectively; in particular, they are nonnegative integers.

Definition 27. The *involution Stanley symmetric function* (for the group K) of $z \in \mathcal{I}_K$ is $\hat{F}^K_z = \lim_{m\to\infty} \hat{\mathfrak{S}}^K_{1^m\times z} = \sum_{w\in\mathcal{A}_K(z)} F_w$.

Let δ_k be the staircase partition $(k-1, k-2, \ldots, 1)$. Theorems 28 and 31 are stabilizations of the identities in Section 3.1.

Theorem 28 ([10], Theorem 3.44)**.** *Suppose $z \in I_K$ is weakly dominant with k 2-cycles, and that the Rothe diagram of $r(z)$ is equivalent via row and column permutations to a skew shape $\delta_m \setminus \mu$. Then $\hat{F}^K_z = \hat{F}^K_{g_k} s_{\delta_m\setminus\mu}$.*

Corollary 29. *Let $p = \lceil \frac{n+1}{2} \rceil$ and $q = \lfloor \frac{n+1}{2} \rfloor$. Then $\hat{F}^{\mathrm{O}(n)}_{w_n} = s_{\delta_p} s_{\delta_q}$ and $\hat{F}^{\mathrm{Sp}(n)}_{w_n} = s^2_{\delta_n}$.*

A partition λ is *strict* if $\lambda_i > \lambda_{i+1}$ for all $i < \ell(\lambda)$. The *shifted Young diagram* of a strict partition λ is $\{(i,j) : 1 \le i \le \ell(\lambda), i \le j < i + \lambda_i\}$. A filling of a shifted shape $\lambda \vdash \ell$ on the alphabet $\{1' < 1 < 2' < 2 < \cdots\}$ is a *marked shifted semistandard tableau* if it is weakly increasing down columns and across rows, no row (resp. column) contains the same primed (resp. unprimed) entry twice, and no primed entry appears on the main diagonal. Such a tableau is *standard* if its entries are $1, 2, \ldots, \ell$ (potentially with primes).

Definition 30. The *Schur P-function* of shifted shape λ is $P_\lambda = \sum_T x^T$, where T runs over marked shifted semistandard tableaux of shape λ.

For simplicity we state the next theorem for the case $K = \mathrm{O}(n)$, but an appropriate analogue holds when $K = \mathrm{Sp}(n)$ as well.

Theorem 31 ([13])**.** *Let $z = (\phi_1\, m{+}1)\cdots(\phi_k\, m{+}k)$ be an I-Grassmannian involution. Associate to z the strict partition λ with $\lambda_i = m - \phi_i + 1$. Then $\hat{F}^{\mathrm{O}(n)}_z = P_\lambda$.*

Theorem 32. *For $z \in \mathcal{I}_K$, the involution Stanley symmetric function $\hat{F}^K_z$ is Schur-P-positive.*

Proof sketch. By stabilizing Theorem 26 and making judicious choices of p, q, one can construct a tree of involutions with root z, leaves some I-Grassmannian involutions, and such that if y is a non-leaf vertex then $\hat{F}^K_y = \sum_{y'} \hat{F}^K_{y'}$ where y' runs over the children of y. This is an analogue of Lascoux and Schützenberger's maximal transition tree; it immediately implies the theorem, and provides an effective algorithm for computing the Schur-P expansion of $\hat{F}^K_z$. (Alternatively, see Theorem 34 below.) □

Schur P-functions appear in several representation-theoretic contexts and in type B and C Schubert calculus, but as yet we have no clear connection between these appearances and our results.

Given any shifted shapes λ and ν, it is easy using Theorem 31 to find an involution y such that $\hat{F}_y := \hat{F}^{\mathrm{O}(n)}_y = P_\lambda P_\nu$. Applying the algorithm described in the proof sketch of Theorem 32 to $\hat{F}_y$ then yields a new Littlewood-Richardson rule for Schur P-functions.

Another consequence of Theorem 32 is a new proof of a conjecture of Stanley, proven using different techniques by Ardila and Serrano [1] and by DeWitt [8].

Theorem 33. *For any $\mu \subseteq \delta_m$, the skew Schur function $s_{\delta_m \backslash \mu}$ is Schur-P-positive.*

Proof sketch. By [10, Theorem 3.1], any 321-avoiding $z \in \mathcal{I}(S_n)$ has a unique atom w, and w is again 321-avoiding. Then $\hat{F}_z = F_w$ is a skew Schur function by [19, Ch. 2], and one can explicitly construct z so that $F_w = s_{\delta_m \backslash \mu}$. Now apply Theorem 32. □

Let g^λ denote the number of marked shifted standard tableaux of shape λ. Writing $\hat{F}_z = \sum_\lambda d_{\lambda,z} P_\lambda$ where $d_{\lambda,z} \geq 0$ by Theorem 32, we then have $|\hat{\mathcal{R}}(z)| = \sum_\lambda d_{\lambda,z} g^\lambda$.

Theorem 34. *Fix $z \in \mathcal{I}(S_n)$. There is a finite set $\mathcal{T}$ of (unmarked) shifted semistandard tableaux such that the* shifted Hecke insertion *of [14] restricts to a bijection*

$$\hat{\mathcal{R}}(z) \to \{(P, Q) : P \in \mathcal{T}, \text{ and } Q \text{ is marked shifted standard of the same shape as } P\}.$$

Moreover, $d_{\lambda,z} = \#\{P \in \mathcal{T} : P \text{ has shape } \lambda\}$.

Theorem 34 also gives a second, purely combinatorial proof of Theorem 32 in the case $K = \mathrm{O}(n)$. We know no analogous insertion algorithm for the case $K = \mathrm{Sp}(n)$.

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